

# An empirical comparison of different risk measures in portfolio optimization

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Risk is one of the important parameters in portfolio optimization problem. Since the introduction of the mean-variance model, variance has become the most common risk measure used by practitioners and researchers in portfolio optimization. However, the mean-variance model relies strictly on the assumptions that assets returns are multivariate normally distributed or investors have a quadratic utility function. Many studies have proposed different risk measures to overcome the drawbacks of variance. The purpose of this paper is to discuss and compare the portfolio compositions and performances of four different portfolio optimization models employing different risk measures, specifically the variance, absolute deviation, minimax and semi-variance. Results of this study show that the minimax model outperforms the other models. The minimax model is appropriate for investors who have a strong downside risk aversion.

**JEL Classifications:** CO2, C61, G11

**Keywords:** Portfolio, optimization, risk measures, variance.

## Introduction

Since the introduction of Markowitz (1952) Mean-Variance (MV) model, variance has become the most common risk measure in portfolio optimization. However, this model relies strictly on the assumptions that the returns of assets are multivariate normally distributed or investor's utility function is quadratic. Nonetheless, in practice these two assumptions do not hold. Many studies such as Brooks and Kat (2002) show that returns from hedge funds are not normally distributed. According to Pratt (1964), quadratic function is very unlikely because it implies increasing absolute risk aversion. Thus, to overcome the limitations of MV model, alternative risk measures such as Mean Absolute Deviation (MAD), minimax (MM) and lower partial moment (LPM) have been proposed. Thus, the objective of this paper is to compare the portfolio compositions and performances of four portfolio optimization models specifically the variance, absolute deviation, minimax and semi-variance as risk measures.

The rest of the paper is structured as follows. The next section discusses the mathematical models, concepts, advantages and disadvantages of using the different risk measures in portfolio optimization namely the absolute deviation, minimax and lower partial moment. Section 3 discusses the empirical results employing the four optimization models mentioned using data of the Malaysia stock market. Section 4 concludes the paper.

## Risk Measures

### Variance

Markowitz (1952) proposed the mean-variance (MV) or Markowitz model by using variance as the measure of risk while mean return as the expected return. Markowitz was the pioneer of the modern portfolio theory. The objective of MV model is to find the weight of assets that will minimize the portfolio variance at a level of required rate of return. This model is a quadratic programming model. The mathematical model is as follows:

$$\text{minimize } \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} x_i x_j$$

$$\begin{aligned} &\text{subject to } \sum_{j=1}^n r_j x_j \geq \rho M_0, \\ &\sum_{j=1}^n x_j = M_0, \\ &0 \leq x_j \leq u_j, j = 1, \dots, n. \end{aligned} \tag{2.1}$$

where  $\sigma_{ij}$  is the covariance between assets  $i$  and  $j$ ,  $x_j$  is the amount invested in asset  $j$ ,  $r_j$  is the expected return of asset  $j$  per period,  $\rho$  is a parameter representing the minimal rate of return required by an investor,  $M_0$  is the total amount of fund and  $u_j$  is the maximum amount of money which can be invested in asset  $j$ .

The Markowitz model is popular because of its simplicity. This model consists of two summary statistics which are mean and variance. Moreover, it is easy to construct the efficient frontier with the combination of return and risk.

Nevertheless, the main disadvantage of Markowitz model is that it is very tedious to calculate  $\frac{n(n+1)}{2}$  covariance of assets to build this model. Furthermore, it is difficult to solve this

quadratic programming model in large scale problem. Investor's perception against risk and distribution of stock prices are also not symmetric around the mean. Commonly, the optimal solution consists of many stocks in small amounts. This will lead to large transaction costs to the investors (Konno and Yamazaki, 1991).

### Mean Absolute Deviation

Konno and Yamazaki (1991) proposed a new model using mean absolute deviation (MAD) as risk measure to overcome the weaknesses of variance. This model is equivalent to Markowitz model if the assets returns are multivariate normally distributed. They formulated the model as follows:

$$\begin{aligned} &\text{minimize } w(x) = E\left[\left|\sum_{j=1}^n R_j x_j - E\left[\sum_{j=1}^n R_j x_j\right]\right|\right] \\ &\text{subject to } \sum_{j=1}^n E[R_j] x_j \geq \rho M_0, \\ &\sum_{j=1}^n x_j = M_0, \\ &0 \leq x_j \leq u_j, j = 1, \dots, n. \end{aligned} \tag{2.2}$$

where  $R_j$  is the return of asset  $j$ ,  $x_j$  is the amount invested in asset  $j$ ,  $\rho$  is a parameter representing the minimal rate of return required by an investor,  $M_0$  is the total amount of fund and  $u_j$  is the maximum amount of money which can be invested in asset  $j$ .

Konno and Yamazaki (1991) assume  $r_{jt}$  be the realization of random variable  $R_j$  during period  $t$  ( $t = 1, 2, \dots, T$ ), then

$$r_j = E[R_j] = \sum_{t=1}^T \frac{r_{jt}}{T}. \tag{2.3}$$

$w(x)$  can be approximated as follows:

$$E\left[\left|\sum_{j=1}^n R_j x_j - E\left[\sum_{j=1}^n R_j x_j\right]\right|\right] = \frac{1}{T} \sum_{t=1}^T \left|\sum_{j=1}^n (r_{jt} - r_j) x_j\right|. \tag{2.4}$$

where  $a_{jt} = r_{jt} - r_j$ , (2.5)

then (2.2) converts to the following model:

$$\begin{aligned}
 & \text{minimize} \quad \sum_{t=1}^T \left| \sum_{j=1}^n a_{jt} x_j \right| / T \\
 & \text{subject to} \quad \sum_{j=1}^n r_j x_j \geq \rho M_0, \\
 & \quad \sum_{j=1}^n x_j = M_0, \\
 & \quad 0 \leq x_j \leq u_j, j = 1, \dots, n.
 \end{aligned} \tag{2.6}$$

Model (2.6) is equivalent to the following linear programming model:

$$\begin{aligned}
 & \text{minimize} \quad \sum_{t=1}^T y_t / T \\
 & \text{subject to} \quad y_t + \sum_{j=1}^n a_{jt} x_j \geq 0, t = 1, \dots, T, \\
 & \quad y_t - \sum_{j=1}^n a_{jt} x_j \geq 0, t = 1, \dots, T, \\
 & \quad \sum_{j=1}^n r_j x_j \geq \rho M_0, \\
 & \quad \sum_{j=1}^n x_j = M_0, \\
 & \quad 0 \leq x_j \leq u_j, j = 1, \dots, n.
 \end{aligned}$$

There is no need to calculate the covariance matrix. Furthermore, it is a linear program. Solving a linear program is much easier and faster than solving a quadratic program. The optimal portfolio consists of at most  $2T + 2$  assets regardless of the size,  $n$  while Markowitz model may consist of  $n$  assets.  $T$  can be used to control the number of assets in the optimal portfolio. As such, MAD can solve a large scale portfolio optimization problem (Konno and Yamazaki, 1991). It is also less sensitive to outliers in the historical data (Byrne and Lee, 2004).

On the other hand, ignoring the covariance matrix can cause greater estimation risk (Simaan, 1997). Moreover, MAD penalizes not only the negative deviations but also the positive deviations. There is no difference between positive deviations and negative deviations. Nevertheless, investors prefer higher positive deviations and avoid lower negative deviations in portfolio return.

### Minimax

Young (1998) proposed minimax (MM) model using minimum return as a measure of risk. The MM model is equivalent to MV model if the assets returns are multivariate normally distributed. MM model is a linear programming model. The minimax model is as follows:

$$\begin{aligned}
 & \max M_p \\
 & \text{subject to} \quad \sum_{j=1}^N w_j y_j - M_p \geq 0, t = 1, \dots, T, \\
 & \quad \sum_{j=1}^N w_j \bar{y}_j \geq G, \\
 & \quad \sum_{j=1}^N w_j \leq W, \\
 & \quad w_j \geq 0, j = 1, \dots, N.
 \end{aligned} \tag{2.8}$$

where  $y_{jt}$  is return on one dollar invested in security  $j$  in time period  $t$ ,  $\bar{y}_j$  is average return on security  $j$ ,  $w_j$  is portfolio allocation to security  $j$ ,  $M_p$  is minimum return on portfolio,  $G$  is the minimum level of return, and  $W$  is the total allocation.

Young (1998) defines  $M_p$  as follows:

$$M_p = \min_t \sum_{j=1}^N w_j y_{jt} \quad (2.9)$$

The model (2.8) is equivalent to the following model:

$$\begin{aligned} \max E &= \sum_{j=1}^N w_j \bar{y}_j \\ \text{subject to} \\ \sum_{j=1}^N w_j \bar{y}_j &\geq H, t = 1, \dots, T, \\ \sum_{j=1}^N w_j &\leq W, \\ w_j &\geq 0, j = 1, \dots, N. \end{aligned}$$

The objective of model (2.10) is to maximize expected return subject to the portfolio return exceed minimum level of return, H.

Young (1998) shows that the minimax model has logical advantages if returns are non-normally distributed and when the investors have a strong absolute aversion to downside risk. In addition, it is a linear program so it can be solved faster than MV model. It can also accommodate more complex model such as including fixed transaction costs constraints.

Because of its objective to minimize maximum loss, minimax is sensitive to outliers in the historical data. Furthermore, the minimax model may not be used if lack the historical data on the past returns or a probabilistic model for future returns (Young, 1998).

### Lower Partial Moment

Lower Partial Moment (LPM) model is also known as downside risk model. Bawa (1975) and Fishburn (1977) generalize the lower partial moment model of degree  $a$  around  $\tau$  is defined as:

$$LPM_a(\tau, R_i) = \int_{-\infty}^{\tau} (\tau - R)^a dF(R) \quad (2.11)$$

where  $dF(R)$  is the cumulative distribution function of the assets returns  $R$ ,  $\tau$  is the target return,  $a$  is the degree of the LPM. The target return can be zero, risk-free rate or expected return. The LPM model is a probability-weighted functions of deviations below some target return.  $\alpha$  degree of lower partial moment can reflect investor utility towards risk with regard to below target return. According to Fishburn (1977),  $a = 1$  suits a risk-neutral investor, risk-seeking ( $0 < a < 1$ ) and risk averse behavior ( $a > 1$ ) with respect to returns below the target  $\tau$ . Semi-variance (SV) is a special case of the LPM when  $a$  is equal to 2 and  $\tau$  is equal to  $E(R)$  (Markowitz, 1959). The semi-variance model is as follows (Konno et al., 2002):

$$\begin{aligned} \text{minimize} \quad & \sum_{t=1}^T p_t z_t^2 \\ \text{subject to} \quad & z_t \geq -\sum (r_j - r_j)x_j, t = 1, 2, \dots, T, \\ & z_t \geq 0, t = 1, 2, \dots, T, \\ & \sum_{j=1}^n x_j = M_0, \\ & 0 \leq x_j \leq u_j, j = 1, \dots, n. \end{aligned} \quad (2.12)$$

where  $x_j$  is the amount invested in asset  $j$ ,  $r_j$  is the expected return of asset  $j$  per period,  $\rho$  is a parameter representing the minimal rate of return required by an investor,  $M_0$  is the total amount of fund and  $u_j$  is the maximum amount of money which can be invested in asset  $j$ ,  $P_t$  is the probability that  $R$  achieves  $r_t$ .

Downside risk is better matches investors' perception about risk than the variance because undesirable downside deviations are separated from desirable upside deviations (Markowitz, 1959). The LPM model penalizes only downside deviations.

Since individual downside risk measures cannot be aggregated to the portfolio risk like the way covariances are aggregated, the computation of portfolio risk is tedious (Grootveld and Hallebach, 1999). Besides that, LPM is sensitive to outliers in the observations that are distant from target return (Byrne and Lee, 2004).

## Empirical Results

Portfolios are developed employing the MV (2.1), MAD (2.7), MM (2.8) and SV (2.12) models in order to compare the portfolio compositions and performances of different optimal portfolios. The data consists of monthly returns of 54 stocks included in the Kuala Lumpur Composite Index (KLCI) from January 2004 until December 2007. The minimum rate of return which is represented by  $\rho$  and  $G$  is set to 1% in this study based on past researches. The probability,  $P_t$  is

set to  $\frac{1}{T}$  for semi-variance model. The portfolio performance is calculated using the reward per risk equation (3.1):

$$\text{Portfolio Performance} = \text{mean return} / \text{risk} \quad (3.1)$$

## Portfolio Performances

The Table 1 shows the summary statistics of the optimal portfolios generated.

TABLE 1. SUMMARY STATISTICS OF OPTIMAL PORTFOLIOS

|             | MV     | MAD    | MM     | SV     |
|-------------|--------|--------|--------|--------|
| Mean Return | 0.0100 | 0.0100 | 0.0205 | 0.0100 |
| Risk        | 0.0181 | 0.0123 | 0.0131 | 0.0124 |
| Performance | 0.5525 | 0.8130 | 1.5649 | 0.8065 |

Notation : MV - Mean Variance, MAD - Mean Absolute Deviation, MM - Minimax, SV - Semi-variance

As shown in table 1, the mean return of MM model (0.0205) is the highest among the four models. The most risky portfolio is the MV model (0.0181) while the less risky portfolio is MAD model (0.0123). The MM model (1.5649) shows the highest performance whereas the MV model (0.5525) gives the lowest performance. It is because the MM model is consistent with expected utility maximization principle with the implied utility function representing an extreme form of risk aversion (Young, 1998). This result is also shown by Biglova et al. (2004) who have studied the performance of portfolio optimization models using nine assets in the German market.

## Portfolio Compositions

Table 2 shows the optimal portfolio compositions of 4 different models.

TABLE 2. PERCENTAGE OF STOCKS IN OPTIMAL PORTFOLIOS

| Stock   | MV<br>(%) | MAD<br>(%) | MM<br>(%) | SV<br>(%) |
|---------|-----------|------------|-----------|-----------|
| Affin   | -         | -          | 9.94      | -         |
| BJToto  | -         | -          | 6.45      | -         |
| BAT     | 23.53     | 16.65      | -         | 18.93     |
| Carlsbg | 6.06      | 4.00       | 6.52      | 5.31      |
| CCM     | 3.95      | 5.79       | -         | 5.68      |
| DiGi    | 4.36      | 5.11       | 3.39      | 3.48      |
| GAB     | 6.99      | -          | -         | 1.52      |
| Guoco   | -         | 0.06       | -         | -         |
| IOICorp | 0.32      | 0.27       | 3.69      | -         |
| Kulim   | 2.66      | 0.39       | -         | -         |
| Litrak  | -         | -          | 5.16      | -         |
| Maybank | 6.50      | 20.53      | -         | 12.24     |
| MMCCorp | -         | -          | 7.61      | -         |
| Mulpha  | -         | -          | -         | 2.06      |
| Bernas  | 3.19      | -          | 15.71     | 2.75      |
| PetGas  | 20.93     | 10.37      | 15.05     | 16.18     |
| Pos     | --        | 6.65       | -         | -         |
| PPB     | 0.05      | 1.17       | 6.09      | 1.22      |
| Puncak  | 2.81      | 5.08       | -         | -         |
| RHBCap  | -         | -          | 2.72      | 6.25      |
| Sarawak | 3.22      | 7.63       | -         | 7.59      |
| Shang   | 1.75      | 1.28       | -         | -         |
| Shell   | 6.22      | 2.45       | 10.66     | 5.39      |
| STAR    | -         | 0.04       | -         | 1.18      |
| TA      | 1.01      | 1.79       | -         | -         |
| TChong  | 0.16      | -          | -         | -         |
| Tenaga  | 0.49      | 4.19       | -         | 7.22      |
| UMW     | 5.82      | 6.54       | -         | 1.19      |
| YTL     | -         | -          | 7.02      | 1.82      |

Notation: MV - Mean Variance, MAD - Mean Absolute Deviation, MM - Minimax, SV - Semi-variance

Results indicate that the portfolios generated by the four risk measures do not differ very much. The difference is on the weight of stocks. According to Byrne and Lee (2004), the difference in weight is probably due to the non-normality displayed by data. The optimal asset allocations of MM model are least like the MV model while the asset allocations of MAD model are very similar to MV model. This result is also shown by Byrne and Lee (2004).

## Conclusion

This paper discusses the theory of risk measures and compares the portfolio optimization models with different risk measures. The result shows that MM model outperforms the other models. As shown by the result of this study, the MV model does not perform as well as other models. As such, the MM model is a better choice for portfolio optimization compared to the other models for it ranks highest in terms of performance. This model is appropriate for investors who have a strong downside risk aversion. Future researches should include more alternative risk measures such as value at risk and conditional value at risk in portfolio optimization.

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